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"It's Understandable Enough, Right?" The Natural Accountability of a Mathematics Lesson

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A fundamental problem for all educational reform efforts is one of specifying in practical terms just what the reform requires of teachers and students. In addressing it, educational designers have introduced the term "accountability" to characterize something missing in classroom discourse. This term, however, held special significance in the writings of the American sociologist Harold Garfinkel. Livingston applied Garfinkel's treatment of accountability to understanding the lived-work of doing mathematics, particularly proving work. We examine an 8th-grade student's presentation during a Japanese geometry lesson as a proof-account. Within it we see elements of both classical and natural accountability placed on display.

> A proof is the discovery of the reasoning unique to that particular proof made available through the material-specific description of it. — Livingston (2008, p. 854)

Within mathematics education there has been strong and sustained advocacy for what might be termed, borrowing a phrase from Sfard and Kieran (2001), the "learning-by-talking" thesis. The thesis, plainly stated, is that mathematical learning is enhanced in the context of interaction that is recognizably and thoughtfully mathematical (e.g., Abrahamson, Gutiérrez, & Baddorf, 2012; Ball & Bass, 2000; Hiebert et al., 1996; Hufferd-Ackles, Fuson, & Sherin, 2004; Lampert, 1990; Lampert & Cobb, 2003; Maher, Davis, & Alston, 1991; Moschkovich, 2008; Romberg & Kaput, 1999; Schoenfeld, 1991; Solomon & Nemirovsky, 2005; Stein, Engle, Smith, & Hughes, 2008). This leaves, however, the difficult question of exactly what we want students and teachers to say and do when they are talking mathematics. We might call this the *specification problem*.

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Michaels, O'Connor, and Resnick (2008) sought to advance the learning-by-talking agenda with a suggestion that talk in mathematics classrooms must not only touch on mathematical topics, but must also be "accountable." This, they argued, entails students displaying an accountability to (a) standards of "respectful and grounded discussion" (p. 286), (b) the factual knowledge of the discipline (e.g., mathematics, science, social studies), and (c) the "standards of reasoning" (p. 286) used therein. The authors' Accountable Talk® proposal is designed as a way of introducing a more "deliberative" (p. 293) form of discourse to the classroom. It represents a possible solution to the specification problem.

It is interesting that Michaels et al. (2008) fixed upon accountability as the missing ingredient in classroom talk. Accountability, as it happens, also holds central place in the writings of the American sociologist Harold Garfinkel. With the publication of *Studies in Ethnomethodology* Garfinkel (1967) laid the groundwork for a new program of inquiry within sociology. Ethnomethodology (EM) took as a founding assumption that "the activities whereby members produce and manage settings of organized everyday affairs are identical with members' procedures for making those settings 'account-able'" (p. 1). This property is what makes social action possible. It is also what holds social institutions together, giving them stability and rendering them orderly. In later writing, Garfinkel (2002) was to make a distinction between *classical* and *natural* accountability. In the everyday sense of the term, one might say, "I am accountable to my boss to be at work on time" or "You are accountable to your clients to provide acceptable service." In this way we are all accountable in our various roles as workers, citizens, parents, teachers, and so on. These are classical forms of accountability.

But there is another kind of accountability that comes before this: We are all accountable, one to the other, to conduct ourselves in ways that are intelligible to others, to act in ways that are sensible. Garfinkel (1967) wrote:

> In exactly the way that persons are members to organized affairs, they are engaged in serious and practical work of detecting, demonstrating, persuading through displays in the ordinary occasions of their interactions the appearances of consistent, coherent, clear, chosen, planful arrangements. (p. 34)

An account, in ordinary parlance, is a report or description of a state of affairs, as when a ledger offers an account of a series of financial transactions. When we engage in social interaction, our conduct stands as "'the document of', as 'pointing to', as 'standing on behalf of' a pre-supposed underlying pattern" (Garfinkel, 1967, p. 78). It gives an account or provides documentary evidence for whatever we might be doing together at the moment—standing in line, waiting for bus, making a purchase, and so on. Garfinkel (1967) stipulated that any social "setting organizes its activities to make its properties as an organized environment of practical activities detectable, countable, recordable, reportable, tell-a-story-about-able, analyzable—in short, *accountable* [emphasis added]" (p. 33). Accountability, in this way, plays an essential role in structuring our social world. Garfinkel elaborated, "Members' accounts, of every sort, in all their logical modes, with all of their uses, and for every method for their assembly are constituent features of the settings they make observable" (p. 8). This is a natural form of accountability.

Natural accountability and the ways in which it is achieved in various work settings became a focus of EM-informed investigations. One of Garfinkel's students, Eric Livingston (1983), undertook a study of mathematicians' work, focusing in particular on "the accountability of the work of proving" (p. 343). His project focused on the published proofs for Gödel's incompleteness theorems. For Livingston, the lived work of mathematics is not to be found in completed proofs, but rather in the settings in which "provers come together and do, for and among each other, the recognizably adequate work of doing recognizably adequate mathematics“ (p. 28). It is here, he argued, that the professional discipline is instructed and in which it is “sustained and renewed, and that it evolves, [and] is revitalized” (p. 28). It has long been recognized that proofs are not just ways of reporting mathematical findings, but also a method for generating new knowledge (cf. Lakatos, 1963; Pólya, 1954). Livingston argued that the work of doing mathematics is integrally tied to a process of discovery and the process of discovery recurs each and every time one works through a proof.

Livingston considered a mathematical proof to have two components: its “material-specific description” (Livingston, 2008, p. 854), the *proof-account* and the “the-practices-of-proving-to-which-that-proof-is-irremediably-tied” (Livingston, 1983, p. 308). Livingston (1987) wrote that the proof-account, i.e. the proof’s “material-specific description” (Livingston, 2008, p. 854) and the “the-practices-of-proving-to-which-that-proof-is-irremediably-tied” (Livingston, 1983, p. 308). The proof-account and the practices of following what the proof-account describes represent what Garfinkel (2002) termed a “Lebenswelt pair” (pp. 187–190).[1] In *Ethnomethodology’s Program*, Garfinkel (2002) adopted the notational convention of flagging certain terms with asterisks to notify the reader that the terms should be read in special way, an EM-informed way.[2] Utilizing the same notation, we refer to the proof-account/practices-of-proof-following pair as a *proof**. A proof*, then, is a Lebenswelt pair consisting of (a) a proof-account and (b) the practical work of following it such that, on any particular occasion of proving, (a) represents a description of (b).[3] It is the natural accountability of the proof that holds the two elements together.

The current article builds upon Livingston’s conceptualization of proof-accounts and proof-followings as naturally accountable pairs and applies it to the study of mathematical problem solving in the classroom. We examine a presentation made by an eighth-grade geometry student in a Japanese classroom. The student is presenting a solution to a posed problem, and the adequacy of this solution must be demonstrated (cf. Lampert, 1990). The student, therefore, must justify his solution, and the justification comes in the form of a logically constructed chain of claims. In this way, the presentation constitutes a kind of proof-account. Our interest is in examining the natural accountability of this presentation and in seeing what this might tell us with respect to the specification problem.

ANALYSIS

> Provers must “find” the proof in the figure. Provers inspect materially definite writings . . . , see through the notational particulars . . . to what they represent, and organize, rearrange and rework such displays to find gestalts of reasoning and practice adequate to a stated theorem. — Livingston (1999, p. 869)

[1]The term *Lebenswelt* comes from Husserl (1970) and can be literally translated as ‘life-world.’ It is “the mundane world of lived experience already existing as a product of the unreflecting cognitions of ordinary actors” (Heritage, 1984, p. 44). Note the discovery that proofs have this paired structure is an ethnomethodological discovery. It was one of the principal findings of Livingston’s thesis (1983).

[2]Examples would include: “read*” (p. 146), “revealed details*” (p. 187), “worksite details*” (p. 187), “naturally accountable details*” (p. 188), “precise description*” (p. 188), and most importantly “order*” (p. 146, FN1).

[3]This reflexive definition closely resembles Garfinkel’s (2002) definition of an “instructed action” (pp. 105–106) and this is not an accident. Both proof*s and “instructed actions,” as Garfinkel conceptualizes them, are Lebenswelt pairs. Indeed, a proof-account might be considered to be a special kind of instruction.

A "Tutorial Problem"

The phenomena of natural accountability are indexical matters. Garfinkel wrote extensively of "occasioned" (Garfinkel, 2002, pp. 204–205) and "indexical" (Heritage, 1984, pp. 142–144) expressions. These are expressions that have a sense wholly dependent upon their settings of production. Common examples are expressions that contain personal deictics such as *you* or *me*, spatial deictics like *here* and *there*, or temporal deictics such as *now* and *then* (consult Chapter 2 in Levinson, 1983). The natural accountability of a proof is context-bound in the precisely the same way—it is lodged within local, endogenous methods of production. Many of the formal methods of social science research discard features of context in the pursuit of generality. To be able to make scientifically warranted claims, the thinking goes, one must isolate oneself from individuating detail and focus instead on commonalities extracted from a multitude of cases. The natural accountability of a proof*, however, cannot be reduced to averages or other measures of central tendency. To preserve these necessary linkages to context, Garfinkel (2002) called for methods of study that rely upon "careful* descriptions" (p. 113) of individual situations studied on a "case* by case*" (p. 173) basis, recalling his instruction that these terms be read and understood in an EM-informed way.

But it is not just the level of detail that separates EM-informed studies from other kinds of inquiries. Garfinkel (2002) wrote that the phenomena of natural accountability "are ubiquitously available to vulgar competence, and elusive—i.e., easy to do and recognize, and intractably difficult to make instructably observable" (p. 174). An ethnomethodologist, then, is faced with the problem of explaining for any "social fact," that is, recognized form of social activity, what it is about the activity, "in its unmistakable, accountable orderliness, that makes it *just this* social fact" (Garfinkel, 2002, p. 250). What we have been describing as a proof* can be thought of as a particular kind of social fact—it is forged in interaction between the prover and proof follower. The task for the ethnomethodologist is one of documenting how this is accomplished in any particular case. Like designers of educational reforms, ethnomethodologists are, in this way, also confronted with a problem of specification. But, where instructional designers are engaged in a principally *prescriptive* exercise, constructing behavioral criteria for successful implementation of a reform effort, ethnomethodologists seek only to illuminate how a social fact comes to be. Theirs is a purely *descriptive* undertaking.

Regarding the reflexive pair we have termed a proof*, Garfinkel (2002) wrote, it "cannot be read off the page no matter now talented a reader the mathematician is" (p. 188). A proof* is an emergent phenomenon, one that is "developingly objective and *developingly accountable*" (Garfinkel, 2002, p. 189). It is for this reason that it is only discoverable both for the participants and for us as external observers. Garfinkel (2002), as a consequence, wrote, "EM's findings are tutorial problems" (p. 115). EM reports are designed to be read in two ways: as a "careful* description" (p. 149) and as instructions for how the things being described can be discovered by the reader.[4] It is in this spirit that the following account is offered. It seeks to document the "revealed details*" (Garfinkel, 2002, p. 187) of the student's presentation as they emerge over time. And in "mis-reading" (p. 149) this description as a set of instructions, it is hoped that the

[4]Garfinkel (2002) referred to this as "praxologizing" (p. 149) the description. For a critical take on Garfinkel's notion of tutorial problems and of "mis-reading" accounts, see Wilson (2003).

reader will come to discover the "gestalts of reasoning and practice" (Livingston, 1999, p. 869) it might entail.[5]

Data Sources

Euclidian geometry courses are the place in the mathematics curriculum in which the rudiments of proof are often first encountered (Herbst & Brach, 2006). The geometry lesson to be described here was of particular interest because of its high degree of interactivity and its heavy reliance on inscription. The lesson was captured as part of the Third International Mathematics and Science Study (TIMSS) Videotape Classroom Study (Hiebert et al., 2003). This study, conducted in two phases (one in 1995 and the other in 1999), was an ambitious effort to document eighth-grade math and science instruction in eight participating countries (Australia, the Czech Republic, Germany, Hong Kong, Japan, the Netherlands, Switzerland, and the United States). It required multiple independent fieldwork teams that eventually taped more than 1,400 classroom lessons conducted in eight different languages.[6] It represents, therefore, one of the largest and most diverse video-based corpora of classroom interaction available. Although access to the TIMSS videocorpus is restricted, a subset of 28 "public release" lessons are available for study. This report focuses on Lesson 744, one of the four public release lessons from Japan. The recording used in this study can be found on the *Talkbank* website.[7] It may be helpful for the reader to download this recording for reference while reading this analysis.

Excerpt 1: "That is what we studied"

```
0:00:54;23    T:  ee:: heekoosen,
 0:00:55;29       uhm  parallel.line
                  "uhm parallel lines"

0:00:57;13    T:  ni- ee:: onaji (0.4) tee⌈hen.⌈(0.3) HHH aruiwa- takasa
 0:01:01;27       at  uhm  same        base              or      height
                  "at uhm the same base, or height"

0:00:59;09    T:                                ⌊((slides index finger
 0:00:59;26        from point A to point B, highlighting the
                   base of the triangle))

0:00:59;28    T:                                ⌊((sweeps index finger
 0:01:00;21       from base toward the vertex of the triangle))

0:01:02;18    T:  no ⌈sankakkee wa (1.1) kono yooni
 0:01:05;04       LK  triangle  Top      this like
                  "such triangles are, like this"

0:01:02;18    T:     ⌊((using computer keyboard, pulls point P first
 0:01:07;24        left and then right along the upper parallel line))
```

[5]Another example of a proof* presented as a tutorial problem can be found in Livingston (1987, Chapters 14–16).

[6]Classes in Hong Kong were mostly conducted in English; classes in Switzerland were conducted in German, French, and Italian. Other languages included Czech, Dutch, and Japanese.

[7]http://talkbank.org/media/ClassBank/TIMSS-Math/Japan-unlinked/744/

```
0:01:06;21    T:  subete- onaji da yo. °to° yuu benkyoo o ⌈shimashita.
 0:01:08;28       all     same  Cop FP  QT  say study   O  did
                  "all of them are the same, that's what we studied."

0:01:08;19    T:                                               ⌊((drops
 0:01:09;01        point "C" on upper line forming triangle ACB))

0:01:09;05    T:  ⌈tatoeba    koko. (1.3) kore.
 0:01:11;09       for.example here        this
                  "For example this here"

0:01:09;05    T:  ⌊((drops points "D", "E" and "F" on upper line
 0:01:16;00        forming three additional triangles))

0:01:18;27    T:  ⌈de kore wa:: (0.6) ee::
 0:01:20;17       and this Top        uhm
                   "And these are uhm"

0:01:18;27    T:  ⌊((brings point P to rest between C and D))

0:01:21;27    T:  ⌈subete::
 0:01:22;12        all
                   "all"

0:01:21;27    T:  ⌊((uses program to compute and display the heights of
                  triangles ADB, APB, ACB, AEB, and AFB))

0:01:22;25    T:  takasa ga onajini naru   kara::
 0:01:24;02       height S  same Nom become because
                  "so their heights end up being the same"

0:01:24;15    T:  menseki ga hitoshiku naru   n   da yo.
 0:01:25;21       area    S  equal     become Nom Cop FP
                  "their areas are the same."

0:01:26;09    T:  to yuu benkyoo o shita n   desu ne.
 0:01:27;09       QT say study   O did   Nom Cop  FP
                  "that is what we studied."
```

The transcript is a tool for cataloging observed action with a recording. A sample can be found in Excerpt 1.[8] Several features of this way of representing talk and visible action are worth noting. As is frequently done in linguistic studies of non-English materials, the transcript is presented in three tiers. The first is a phonetic transcription of the teacher's (T) talk presented in transliterated Japanese (*Roma-ji*). Because timing and prosodic features of delivery (e.g., intonation, volume, tempo) are crucially important to meaning construction, the talk is transcribed employing Conversation Analytic conventions.[9] Japanese employs a subject–object–verb word order.

[8]The transcripts were prepared by one of the authors, a native speaker of Japanese.

[9]The full set of conventions is described in Jefferson (2004). In summary, numbers enclosed in parentheses represent periods of silence measured to a tenth of a second. Brackets are used to mark talk or other forms of action delivered in overlap. Use of standard punctuation marks such as periods and question marks denotes delivery with falling or rising intonation resembling that ordinarily heard at the end of a sentence (or question). Colons are used to display sound

The second tier provides a literal translation ordered by Japanese syntax. Thus, lexical items are replaced, one by one, by their English counterparts, and Japanese function words, which have no direct English translation, are represented using special symbols.[10] For example, the teacher's "takasa ga onajini naru kara" [0:01:22;25] is parsed as "height <subject marker> same to become because." *Onajini*, a conjugation of *onaji* and *ni*, is treated as separate morphemes. The resulting literal translation is presented colloquially in the third tier as, "so their heights end up being the same." From the colloquial gloss, however, it is not possible to see how the talk is precisely coordinated with other concurrent action, and so both literal and colloquial translations are needed here.

Livingston (1983) reported, "A prover uses his embodied presence to the blackboard and to the audience to achieve the exhibited precision of his work and talk" (p. 3). Our analysis, therefore, cannot and should not be confined to the participants' talk. Indeed, starting from McNeill's (1979) early study of gesture use among mathematicians and continuing to more recent work (see, e.g., Alibali & Nathan, 2012; Greiffenhagen & Sharrock, 2011; Hall & Nemirovsky, 2012; Núñez, 2008; Radford, Edwards, & Arzello, 2009), there is a growing consensus that mathematical sense-making needs to be studied as a multimodal matter. To this end, also included in the transcript are annotations describing various bodily actions conducted in concert with the talk. For example, while the teacher states "at uhm the same base, or height" [0:00:57;13] he points two times toward the computer screen. These pointing actions are durational events that occur in overlap with his talk. For each item in the transcript (either annotation or attributed utterance), its start and ending frame numbers (expressed in hours, minutes, seconds, and frames) are provided in the left-hand column. The precise timing of overlap of talk and action is also marked within the transcript. We see, for instance, the teacher's first point to the screen (at [0:00:59;09]) occurs at the start of the second syllable of *teehen* (base), and it is this temporal coordination that gives the manual action its recognizable sense.

As part of the described lesson, two students presented alternative problem solutions at the board. Our analysis focuses on the first of these presentations. Before launching into it, however, we first explore how the problem was originally posed to the class.

Review of Prior Lesson: "That Is What We Studied"

As the lesson begins, the class is called to order by one of the students. Students are directed to stand and bow to the teacher, who is positioned at the front of the room. The teacher offers a bow in return. There are many things about this scene—how the room is furnished, the positioning of its members, how participation is structured—that are immediately recognizable to all, even those of us who have never visited a Japanese classroom. We experience a certain sense of familiarity with this as a certain kind of institutional setting.

stretching. Text enclosed between degree signs represents talk delivered at diminished volume. Annotations are enclosed in double parentheses and italicized.

[10]The following symbols are employed for Japanese function words: Cop (copula), FP (final particle), LK (linking marker), Neg (negative), Nom (nominalizer), Top (topic marker), O (object), Q (question particle), QT (quotative particle), and S (subject marker).

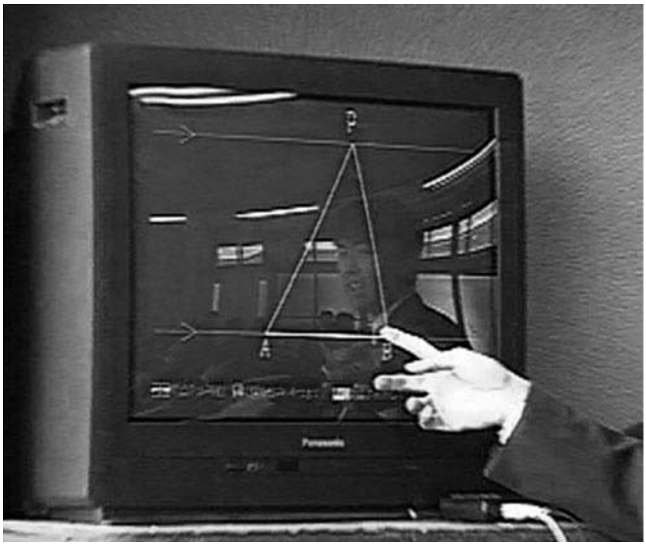

FIGURE 1 The teacher uses the computer display to review the theorem. [00:00:59;21].

The teacher switches on a computer monitor positioned to the left of the board and visible to the class. Shown on the screen are two horizontal lines presented as parallel. Interposed between the lines is a triangle, APB, with its base (labeled AB) situated on the lower of the two lines and its apex, P, located on the upper (see Figure 1). Looking to the class, the teacher asks, "Uhm do you remember what we did in the previous class?" [0:00:28;05], to which one student answers, somewhat unhelpfully, "Math" [0:00:30;29]. The instructor reframes his question and addresses it to a particular student, "Naito what kind of thing did we do?" [0:00:33;16]. The student is uncertain how to respond, however, [0:00:35;23] and so the teacher walks toward the computer monitor and suggests, "This, we studied" [0:00:44;06]. With this prompting, the student articulates the gist of the previous lesson as "Triangles that exist at the place of parallel lines have the same area" [0:00:48;16].

This summary is succinct. To "exist at the place of parallel lines" serves as a gloss for something previously discussed and its understandability rests on prior, presumably shared, understandings. The teacher ratifies this summarization [0:00:53;13] and proceeds to show how it relates to the figure on the computer monitor.

Gesturally the teacher indicates the base (*teehen*) and height (*takasa*) of the triangle APB (see Figure 1). Note in the excerpt how his pointing gestures are carefully coordinated with his enunciation of these two geometric terms. Using the computer to drag the point P along the upper line (see Figure 2), he illustrates a family of triangles having the same base (AB) and, because all have their apex on the upper line, the same height. "Such triangles," he reports, "all of them are the same" [0:01:06;21]. They are what Ms. Naito had previously described as "triangles at the place of parallel lines" and, by the theorem just articulated, they all have the same area.

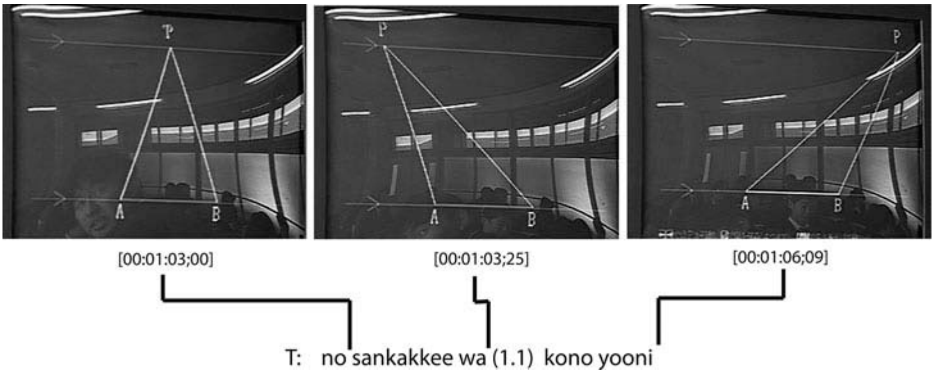

FIGURE 2 The teacher demonstrates, “Such triangles are like this . . .”.

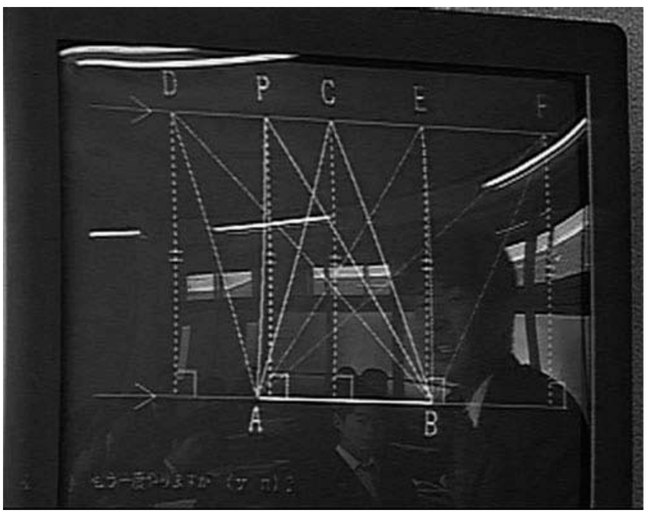

FIGURE 3 Computer-generated triangles illustrating the theorem. [00:01:23;10].

That would seem to settle the matter, but the teacher continues with his review. He drops a series of additional points (C, D, E, and F) on the upper line of the computer-based figure each time leaving behind a formed triangle. He then uses the program to display the heights for each of the triangles drawn (see Figure 3) and restates the theorem as, “their heights end up being the same and so, their areas are the same” [0:01:22;25] and summarizes, “This is what we studied” [0:01:26;09]. He then concludes, “based on this, we will continue our study today” [0:01:28;06]. The computer-generated illustration will remain on the screen as a resource for the duration of the exercise. It thus becomes a locally reference-able, concrete representation of the theorem previously discussed. It serves, in effect, as an account of their prior work.

Excerpt 2a: Formulating the problem.

```
0:01:52;17    T: ⌈Ee::::
 0:01:53;08       uhm
                 "uh:::m"

0:01:52;17    T: ⌊((consults notes on his desk))
 0:01:53;16

0:01:56;10    T: ((using large drafting triangle as a straight edge,
 0:02:00;15       draws line on chalkboard))

0:02:02;11    T:  ((draws second line not parallel to the first))
 0:02:03;16

0:02:06;02    T:  ((connects the two lines with a bent line))
 0:02:09;07

0:02:14;11    T:  ee::: ima kotchi    ni desu ne,
 0:02:15;14       uhm   now this.side at Cop  FP
                  "uhm now on this side"

0:02:16;05    T: ((gazes at one student in classroom))
 0:02:17;28

0:02:17;28    T:  Chiba   kun no ⌈tochi ga arimasu.
 0:02:19;00       Surname Mr  LK  land  S  exist
                  "there is Mr. Chiba's land"

0:02:18;23    T:                ⌊((writes kanji character for
 0:02:24;03        student's name on left side of diagram))

0:02:23;01    T:  Hai  Chiba   kun no tochi desu. (1.3) ne?
 0:02:25;16       okay surname Mr  LK land  Cop         FP
                  "Okay. It's Mr. Chiba's land. (1.3) Okay?"

0:02:26;07    T:  Kotchi (0.5) ⌈Bando   kun no tochi desu ne.
 0:02:27;19       this.side     surname Mr  LK land  Cop  FP
                  "This side is Mr. Bando's land, right."

0:02:27;12    T:              ⌊((writes kanji character for another
 0:02:32;01        student's name on right side of diagram))

0:02:32;11    T:  Bando  kun.
 0:02:33;03       surname Mr
                  "Mr. Bando."

0:02:34;02    T:  Kooyuu    tochi ga atta  to shimasu.
 0:02:35;03       this.kind land  S  exist QT do
                  "Suppose there is land like this."

0:02:36;25    T: ⌈DE (0.5) ii   kai  (       )?
 0:02:38;07       And      good Q
                  "And, okay (                  )?"

0:02:36;25    T: ⌊((extends pointer taken from his pocket))
 0:02:38;02
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```
0:02:38;29    T:  hai (0.2) de kono ⌈(0.3) futari no (0.7)
 0:02:44;03       okay      and these       two    LK

                  ah::: kyookaisen ga koo      magatteru n desu ne.
                  uhm   borderline S  this.way bent      Nom Cop FP

                  "Yes.  And the borderline between these two is bent
                  like this."

0:02:40;05    T:                   ⌊((highlights the bent line using
 0:02:41;10       the pointer tip highlights))

0:02:44;27    T:  Massuguni shitai  n   desu.
 0:02:45;17       straight  do:want Nom Cop
                  "We want to make it straight."
```

Problem Presentation

Formulating the problem. Following the review of the previous lesson, the teacher moves to the chalkboard. Consulting his lesson plan on the desk and using a large drafting triangle as a straightedge, he produces the drawing shown in Figure 4. As he explains, the bent line in the middle represents the line dividing the properties of two neighbors. The property to the left of the crooked line is labeled as belonging to "Chiba;" the property to the right as belonging to "Bando." These reference two students in the class selected on the spot as stand-in landowners

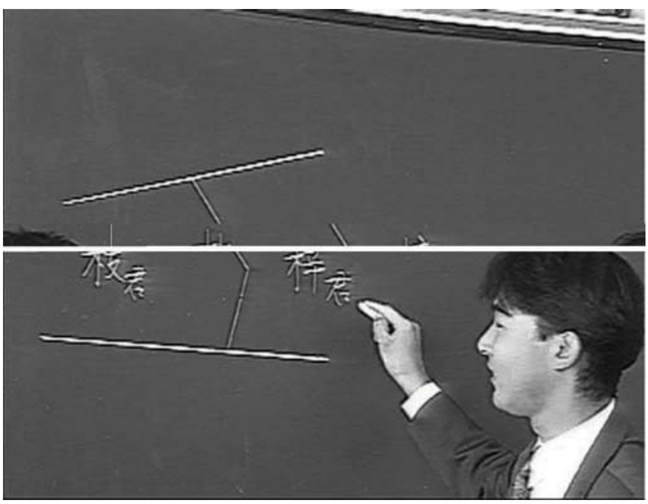

FIGURE 4 The exercise diagram depicting Mr. Chiba and Mr. Bando's initial property holdings. [00:02:32;04].

for the purposes of the exercise.[11] The crux of the problem is to find a way of equitably redrawing the property line making it straight, but preserving the areas of the previous land holdings ("And the borderline between these two is bent like this. We want to make it straight." [0:02:39;17]). The project, then, is to replace the bent property line with a straight one. The crucial issue revolves around where to place the new line.

Excerpt 2b: Exploring the problem space, informally.

```
0:02:47;00   T:  Chiba   kun,
 0:02:47;06      surname Mr
                 "Mr. Chiba,"

0:02:47;20   C:  hai
 0:02:47;26      yes
                 "Yes"

0:02:48;12   T:  kono hen ⌈de ii kana?
 0:02:49;16      this around good FP
                 "Around here would be okay?"

0:02:48;23   T:           ⌊((rests pointer on board to the right of
 0:02:52;24      the bent property line and grins))

0:02:51;07   T:  hai
 0:02:51;13      yes
                 "Yes"

0:02:51;19   T:  ii   su ka? =
 0:02:52;02      good Cop Q
                 "Is it okay?"

0:02:52;03   Ss: = ⌈((laughing))
 0:02:52;24

0:02:52;03   T:  = ⌊((laughing))
 0:02:52;24

0:02:53;02   T:  ja   kyoo  no benkyoo owarimasu ne. =
 0:02:54;00      then today LK study   finish    FP
                 "Then today's lesson is over."

0:02:54;03   Ss: = ⌈((laughing))
 0:02:55;18

0:02:54;03   T:  = ⌊((laughing))
 0:02:55;18

0:02:56;27   T:  Bando    kun kono hen de ⌈ii kana?
 0:02:57;22      surename Mr. this around good FP
                 "Mr. Bando's, would around here be okay?"

0:02:57;17   T:                           ⌊((rests pointer on board to
 0:03:01;04       the left of the bent property line and again grins))
```

[11]These pseudonyms come from the transcript in the TIMSS database.

```
0:02:58;10    B:  a:::::::n (°da⌈me°)
 0:03:00;07       uhm        no.good
                  "uh:::::::m (no)"

0:03:00;17    T:               ⌊Dame. (0.3) ⌈Dono hen  ga ii desu ka?
 0:03:00;21                     no.good    which around S good Cop Q

                                "No?  Around where would be good?"

0:02:57;17    T:                            ⌊((slides pointer to the
 0:03:05;14       right))

0:03:03;10    S?: (°                    °)

0:03:05;14    B:  ya
 0:03:05;18

0:03:06;09    T:  un? Moo chotto  kotchi? (0.7) Dono hen kana?
 0:03:08;11       huh more little this.side    which around FP
                  "What? A little more to this side? Around where?"

0:03:09;25    B:  °motto kotchi.° =
 0:03:10;10        more this.side
                  "More this way"

0:03:10:10    T:  = motto ⌈kotchi
 0:03:10;24         more this.side
                  "More this way"

0:03:10;19    B:          ⌊⌊motto motto motto (0.0) sono hen° =
 0:03:12;01                 more  more  more       that around
                            "°More more more (0.0) around there°"

0:03:12;01    Ss: = ⌈((laughing)) =
 0:03:13;22

0:03:12;01    T:  = ⌊((laughing)) =
 0:03:13;22

0:03:13;22    T:  = Chiba   kun kono hen de ii  desu ka? =
 0:03:15;00         surename Mr this around good Cop Q
                    "Mr. Chiba, would around here be okay with you?"

0:03:14;21    C:  = Dame   desu.
 0:03:14;28         no.good Cop
                    "No good"

0:03:15;05    T:  Dame    desu ka. =
 0:03:15;26       no.good Cop Q
                  "No good"
```

Exploring the problem space, informally. Consulting the two ersatz landowners, the teacher solicits suggestions for where the new property line might be placed. He uses his pointer to demonstrate alternatives by laying it flat over the exercise diagram. He positions the

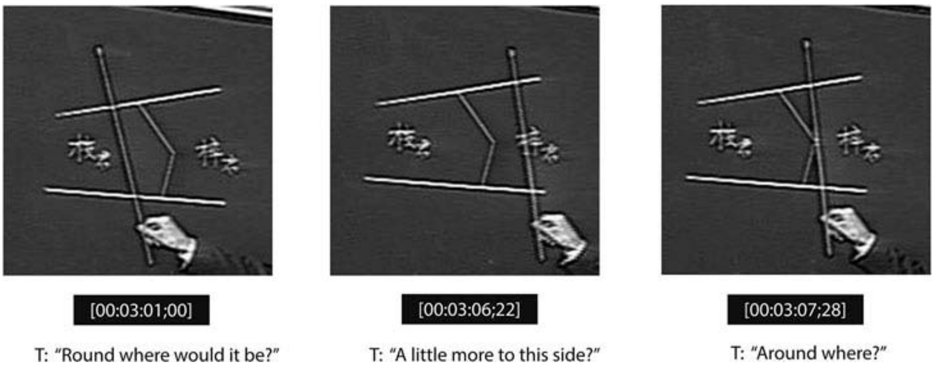

FIGURE 5 The teacher provides options for the revised property line to Mr. Bando.

pointer to the right of the bent property line and asks Chiba, "Around here would be okay?" [0:02:48;10]. The student responds in the affirmative and the class laughs. The teacher jokes, "Then today's lesson is over" [0:02:53;02]. It is understood by all, and displayed as such, that positioning the property line in this way would unfairly favor Chiba over his fictional neighbor Bando. The teacher offers a similar set of options to Bando. Beginning slightly to the left of the bent line, he slides the pointer first to the right and then to the far left (see Figure 5) before finding a configuration acceptable to the second student cum land holder. Bando's selection is also greeted with laughter. This bit of role-play rests on shared understandings of what it means to own land and the importance, having once acquired it, of not being cheated out of it. Formulating it as a property dispute works in this way both to motivate what might otherwise seem to be an abstract geometric theorem and to illustrate its potential power for solving practical problems.

The previously given demonstrations were designed to show that the positioning of the new property line was not arbitrary and that both property holders had to be treated fairly. The teacher now invites a third student to propose a replacement property line, presumably one with a better, geometrically based justification. Note that turning over the pointer to a student at this phase of the exercise represents a bit of a gamble. Should the student come up with the solution being sought, there would be nothing left for the class to do and the exercise would indeed collapse.[12] Her proposal involves constructing a pair of vertical lines. The first will connect the open endpoints of the two line segments representing the bent property line (see Figure 4), forming a triangle. The second line, drawn parallel to the first, passes through the inflection point of the bent property line. Her proposal is that a third line positioned midway between the two parallel lines might represent an equitable solution to the posed problem. Indeed, her approach might superficially resemble the summarized theorem, incorporating both a pair of parallel lines and a triangle positioned between them. Her proposal, however, sacrifices the power of the theorem,

[12]Or, maybe not. Were this to occur, it is likely that the teacher would challenge the class to develop an alternative solution, there being another available.

failing to exploit its ability to predict the area of conforming triangles. The teacher now turns the problem over to the full class. He unfurls a banner that reads, "Without changing the area, how to change the shape." By now employing the term "area" (*menseki*) instead of "land" (*tochi*), the teacher orients the students to a reconsideration of the summarized theorem.

Several aspects of this problem are unspoken, but understood by all. These are taken-for-granted features, unremarked and unremarkable, of how one solves a mathematical word problem. For example, though stated as a property line negotiation, the solution to the problem has nothing to do with property lines per se, and this is understood from the outset. It is also worth noting that no one proposes to solve the problem by measurement. First, given the way in which the two properties are represented in the exercise diagram, it is not possible to estimate the areas of Chiba and Bando's land. And, although the problem diagram was drawn very carefully, the angles and areas shown there are understood to be inexact. Indeed, the areas and even the shapes of the properties are arbitrarily represented. As Livingston (1983) noted:

> The solution must not only hold for the case represented on the board, but for a broad class of cases not depicted. The problem of characterizing the mathematical object is the problem of determining what kind of object a mathematician refers to when he proves theorems about them; what kind of objects are circles qua circles and angles qua angles. (p. 10)

For this reason, though the assignment calls for making a new straight line to divide the properties, simply producing such a line will not satisfy the assignment. It is incumbent upon the problem solver to show *how* the solution works, not only for the case shown in the exercise diagram, but for others as well (cf. Herbst, 2005). Moreover, it must be *demonstrated* how the line relates to and satisfies the various requirements of the problem. A sketched property line is not a mathematically adequate solution, therefore—it must include a justification for *why* the line satisfies these requirements.

The teacher first directs the students to work on the problem individually [0:03:56;23]. After several minutes of deskwork he invites them to avail themselves of other resources in the room—the two instructors, hint cards at the classroom front, their fellow students. During this period of working informally, the teacher roams the room interacting with students, individually and in small groups. This method of collaborative problem solving is one commonly employed in Japanese mathematics classrooms (Shimizu, 2002).

The crux of the posed exercise is one of discovering how the reviewed theorem relates to the property line issue or, stated visually, of transposing the theorem illustration onto the exercise diagram. The teacher employs different ways of helping the students to achieve this. While working with a group of three students at the board, he instructs them first to look at the theorem illustration on the TV monitor (Figure 6a) and then directs them to reexamine the exercise figure, but with their heads cocked to one side (Figure 6b). We see in this an instructing of a certain way of looking, carried out in an embodied way. Later, to make visible the possibility of a triangle in the problem figure, he places his pointer on the board connecting the points where the crooked property line intersects the upper and lower boundaries of the two properties. This too is an embodied form of instruction. Two of the students at the board will later be called upon to present their solutions to the class.

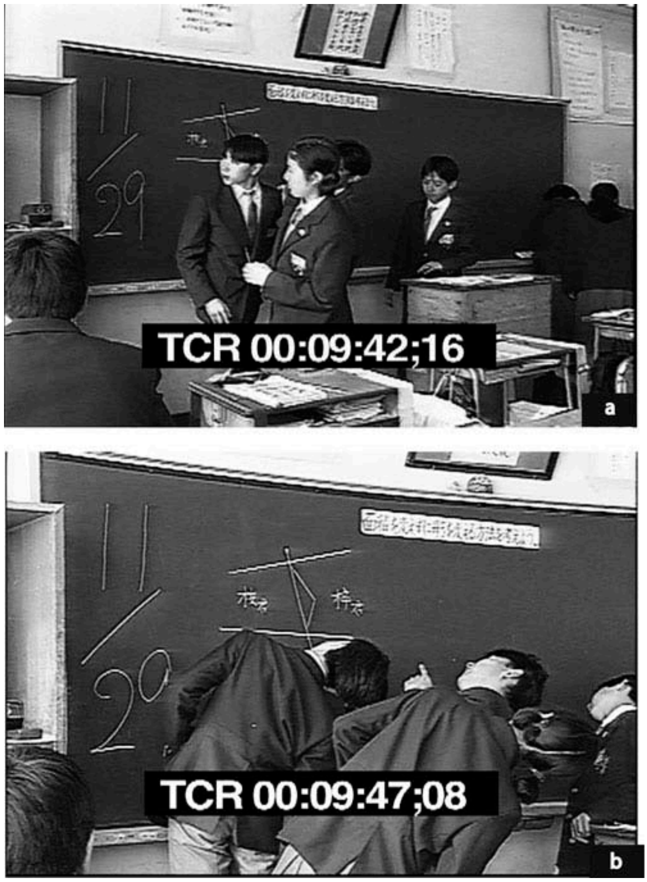

FIGURE 6 The teacher instructing two students on how to see the theorem illustration on the computer monitor (off camera to the left) in the exercise diagram.

In preparation for the whole-class discussion to come, the teacher directs two students to make copies of the exercise diagram on the lower region of the chalkboard, providing them with colored chalk and straightedges [0:16:24;16–0:17:00;16]. The two solution figures they produce constitute predrawn resources for later whole-class discussion. They are not identical. The one on the left includes a red line; the solution figure on the right contains a yellow line. The positions of these two lines are highlighted in Figure 7.

After approximately 10 min, the class members are asked to return to their seats. The teacher offers the pointer to one of the members of the class, a student whom we know here as "Mr. Manabe."

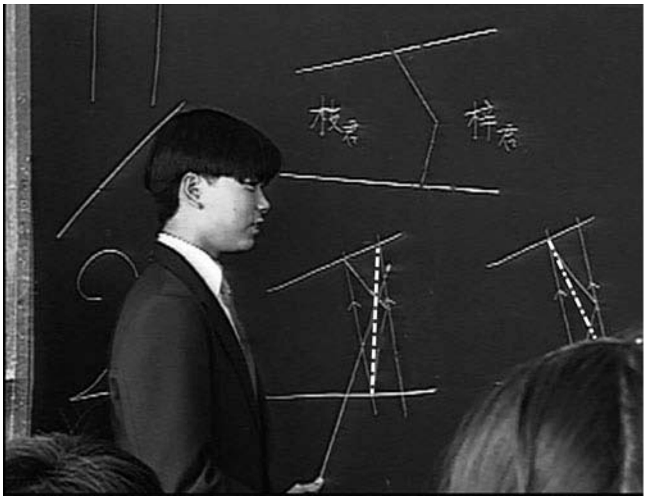

FIGURE 7 Mr. Manabe at the board about to present his solution. The two, pre-drawn solution figures can be seen before him on the board. The red and yellow chalked lines, in the left and right solution figures respectively, are highlighted using dashed lines.

Excerpt 3a: A rough start.

```
0:19:47;07    M:  kore wa desu ne::, (1.2) uhh
 0:19:49;15       this Top Cop FP
                  "This i:::s, (1.2) uhh"

0:19:50;11    M:  mazu  wa  ⌈sankakkee o tsukuru n  ⌈desu yo.
 0:19:51;22       first Top triangle   O make   Nom Cop  FP
                  "First, we make a triangle."

0:19:50;20    M:            ⌊((points toward exercise diagram above))

0:19:51;18    M:                                    ⌊((points toward
                  solution figure below))

0:19:52;22    Ss: ((laughter))
 0:19:53;23

0:19:53;08    S:  nani itte n   da  yo.
 0:19:54;07       what say  Nom Cop FP
                  "What are you saying?"
```

```
0:19:54;20    M:  omee ⌈urusee ⌈yo.
 0:19:55;08       you   noisy   FP
                  "You are noisy."

0:19:54;04    M:      ⌊((waves pointer toward speaker))
 0:19:55:08

0:19:55;00    S?:                 ⌊heekoosen     o ⌈(hiku n   da  yo)
 0:19:55;21                        parallel.line O  draw Nom Cop FP
                                  "draw a parallel line"

00:19:55;14   Ss:                                 ⌊((laughter))
 00:19:56;27

0:19:56;04    M:  sankakee o tsukuru n ⌈desu yo.
 0:19:57;09       triangle O  make   Nom Cop FP
                  "Make a triangle."

0:19:56;28    M:                         ⌊((directs eyes to camera
 0:19:57;09        and nods))

0:19:57;16    M:  sorede:: (0.2) koko ⌈ni::
 0:19:59;08       then           here  at
                  "Then here (        )"

0:19:58;20    M:                        ⌊((positions tip of pointer at
                  which the crooked line intersects the upper property
                  boundary))

0:19:58;26    S?:                       ⌊kamera mesen da.
 0:19:59;20                              camera gaze  Cop
                                        "Gazing at the camera."


0:19:59;19    M:  ⌈sono (     ) no::
 0:20:01;07        that         LK
                   "that (      )'s"

0:19:59;19    M:  ⌊((traces line beginning at the point the crooked
 0:20:00;22        property line intersects the upper property boundary
                  and proceeding to the point where the crooked
                  property line intersects the lower boundary))
```


```
0:19:59;19    M:  ⌈sono (     ) no::
 0:20:01;07        that         LK
                   "that (      )'s"

0:19:59;19    M:  ⌊((traces line beginning at the point the crooked
 0:20:00;22        property line intersects the upper property boundary
                  and proceeding to the point where the crooked
                  property line intersects the lower boundary))
```


```
0:20:01;25    S?: yari sugi:
 0:20:02;20       do   too.much
                  "Doing too much."
```

An Emerging Proof-Account

A rough start. As noted earlier, merely coming up with a new property line will not suffice as a solution for this exercise; the exercise requires producing some sort of geometric justification. Advancing to the board Mr. Manabe (M) utilizes the predrawn solution figure on the left, the one containing the red line (see Figure 7). The thesis to be developed is that the red line satisfies the spoken and unspoken requirements of the posed problem. His task, then, is one of deriving the line from the problem particulars. He starts with "This is" [0:19:47;07]. Parts of his opening statement (*kore wa desu ne::*), however, consist of function words that do not translate directly in English.

As spoken, it is an appropriate way to launch a proof-account, putting his audience on notice of what he is about to do. But then he hesitates.

Knowing where to start a proof-account is key to the enterprise. Turning to the solution figure on the left, he then offers, "First, we make a triangle" [0:19:50;11]. Mr. Manabe waves the pointer, first toward the exercise diagram (seen above on the board in Figure 7) and then to the solution figure (seen below in Figure 7). His gesture produces one as the starting point and the other as the destination, projecting the trajectory of the presentation to come. It remains, however, for Mr. Manabe to explain just how the original diagram was elaborated to produce the solution figure and how these elaborations were related to the theorem reviewed by the teacher.

Unfortunately, this start is not well received by his classmates. After a brief delay, the class breaks into laughter and one student shouts, "What are you saying?" [0:19:53;08]. An unidentified student suggests that Mr. Manabe should start by constructing a parallel line. Yet another accuses Manabe of playing to the camera [0:19:58;26]. In contrast to the orderly proposing of counter examples described by Lakatos (1963) in his imaginary mathematics seminar, the assessments offered by Mr. Manabe's classmates would appear to be less mathematically focused and plainly less helpful.

Why might Mr. Manabe's proposal to "make a triangle" be greeted with derision? The theorem introduced in an earlier lesson does involve triangles, but triangles of a special type—"triangles that exist at the place of parallel lines." To utilize the theorem to make claims about areas, triangles must be of this special type, that is, they need to be defined in terms of certain, specified parallel lines. So, to proceed, we must have both kinds of things—a pair of parallel lines and some conforming triangles. Addressing his detractors, Mr. Manabe says only "You are noisy" [0:19:54;04] and reiterates "Make a triangle" [0:19:56;04].

Excerpt 3b: A key constructed line.

```
0:19:59;19    M: ⌈sono (        ) no::
 0:20:01;07       that            LK
                  "that (         )'s"

0:19:59;19    M: ⌊((traces line beginning at the point the crooked
 0:20:00;22       property line intersects the upper property boundary
                 and proceeding to the point where the crooked
                 property line intersects the lower boundary))
```

```
0:20:01;25    S?: yari sugi:
 0:20:02;20       do   too.much
                  "Doing too much."

0:20:01;22    M:  ((waves pointer tip above the point where the right
 0:20:04;25       parallel line intersects the upper property boundary
                  while carefully studying that region of the figure))

0:20:02;26    M:  hee hee heekooni::
 0:20:04;11               parallel.in
                  "para- para- in parallel"

0:20:04;25    M:  sen  o  kotchi    ⌈ni mo hiite::,
 0:20:04;11       line O  this.side at too draw
                  "draw a line on this side as well, and uhm"

0:20:05;01    M:                    ⌊((slides tip of pointer along the
 0:20:05;23        right parallel line))
```

A key constructed line. What Mr. Manabe does now is trace the line going from the point at which the crooked property line intersects the upper property bound to the point where the crooked line intersects the lower property bound. With his face turned to the board, Mr. Manabe says something difficult to hear [0:19:59;19] while tracing this constructed line. As seen in Figure 7, the indicated line does indeed "make a triangle." He draws here upon the time-honored practice in geometry of *construction*. Geometric construction entails adding new elements with specified geometric properties (cf. Livingston, 1987, 1999; Stahl, 2013).

Instead of building up the necessary pieces as he goes along, which would arguably be a little easier to follow, Mr. Manabe must locate the constructions within a precompleted representation. Generally speaking, the construction needs to progress from simpler structures (e.g., lines, points) to more complex (e.g., triangles, parallel lines). So, though it might have been clearer had he started by explaining how the line was constructed and, then, noting that it formed a triangle, what he had said was geometrically correct, though it may have raised difficulties for his classmates struggling to follow his presentation. We do not hear him reference the line explicitly, but its determination is made clear through, and only through, his tracing gesture.

He now employs the constructed line to make a second. He directs "draw a parallel line on this side as well" [0:20:05;01]. The first constructed line defines a triangle by joining the two segments of the crooked property line *and* it serves as the basis for constructing a pair of parallel lines. So we find our way back both to Mr. Manabe's initial suggestion that we start with a triangle and the suggestion from the audience that we start with a pair of parallel lines. As with the construction of the first line, the determination of where the line is to be drawn hinges critically on his gesture with the pointer. He traces the parallel line seen on the right in Figure 7. This line passes through the apex of the triangle just developed, as is described in the section to follow.

Excerpt 3c: Locating the first triangle.

```
0:20:04;25    M:  sen  o  kotchi    ⌈ni mo hiite::,
 0:20:04;11       line O  this.side at too draw
                  "draw a line on this side as well, and uhm"

0:20:05;01    M:                    ⌊((slides tip of pointer along the
 0:20:05;23        right parallel line))

0:20:07;11    M:  de   kotchi    wa (0.2) ⌈TEEHEN to suru  n desu yo.
 0:20:09;12       then this.side Top       base  as make Nom Cop  FP
                  "then make this side the base."

0:20:08;17    M:                          ⌊((taps the pointer at points
 0:20:09;18       in which the left parallel line intersects the upper
                  and lower boundaries respectively))

0:20:10;10    M:  teehen ⌈to. (0.8) koko o.
 0:20:11;24       base    as        here O
                  "as the base. (0.8) Here."

0:20:10;12    M:        ⌊((taps at the middle of the l. parallel
 0:20:10;12        line))

0:20:12;08    S?: °un°
 0:20:12;11       uh.huh
                  "uh huh"

0:20:13;02    M:  de  takasa! (0.8) ⌈ni shite::,
 0:20:13;02       and height         as make
                  "and make it the height, and uhm"

0:20:14;19    M:                    ⌊((taps at the points in which
 0:20:15;10       the r. and l. parallel lines intersect the upper
                  property boundary respectively))

0:20:15;17    M:  ⌈kono sankakkee to::
 0:20:17;04       this triangle  and
                  "this triangle a::nd"

0:20:15;17    M:  ⌊((forms a triangle by tracing the crooked property
 0:20:17;27       line from its point of intersection with the upper
                  property line through its apex down to its point of
                  intersection with the lower property line and back to
                  the upper property line — see Figure 8a))
```

Locating the first triangle. At this point in the derivation, Mr. Manabe supplies a bit of geometric terminology. Returning to the first constructed line, he states, "then make this side the base" [0:20:07;11]. He now notes that the distance between the base and the just constructed parallel line represents the height of the triangle [0:20:13;02]. Once again, the specification of this distance, of the *it* in "and make *it* the height" is done gesturally by touching the first and second constructed lines. Establishing the height of the triangle, of course, is crucial for his subsequent use of the theorem. These are the parameters of the structure he now labels ("This triangle" [0:20:15;17]) while tracing the two segments of the crooked property line and the previously specified base (see Figure 8a).

The terms base (*teehan*) and height (*takasa*) had both been introduced earlier by the teacher. We see in the emerging proof* evidence of both classical and natural accountability. Mr. Manabe, for instance, displays an orientation to classical accountability in his adoption of a geometer's nomenclature. But natural accountability is also reflected in the myriad ways in which his talk and manual action display an orientation to an intrinsic and developing logic.

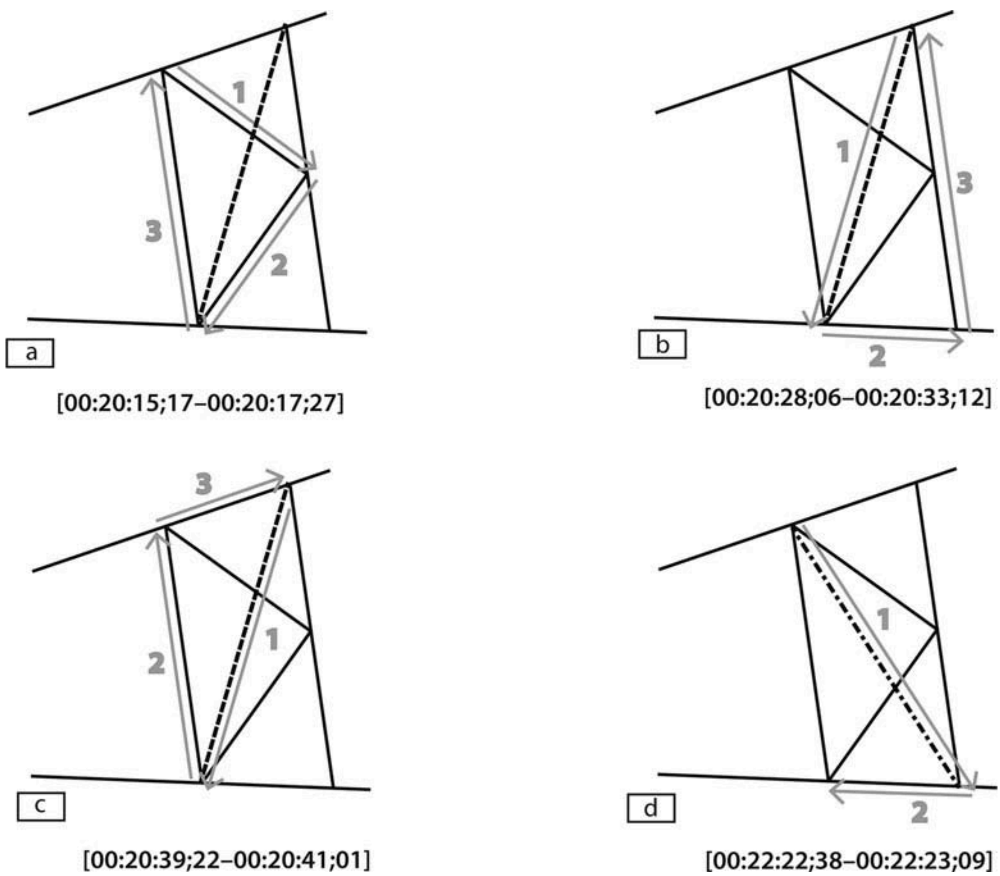

FIGURE 8 Graphic representation of four demonstrations produced at the board, the numbered lines portraying the sequence of strokes traced by the pointer. These include: (a) Mr. Manabe's demonstration of the reference triangle on the leftmost solution figure, (b) an incorrect triangle produced by Mr. Manabe, (c) a conforming triangle, suggested by the teacher and subsequently demonstrated by Mr. Manabe, and (d) an alternate solution demonstrated by Ms. Ikeda on the second solution figure.

Excerpt 3d: Locating the second triangle.

```
0:20:17;16    S:  °sore ga takasa tte⌈okashii n (    )°
 0:20:18;23        that S  height QT  wrong  Nom
                   "Isn't it wrong to call that height?"

0:20:18;10    M:                       ⌊ne!
 0:20:18;15                             FP
                                        "Right!"

0:20:20;06    M:  ⌈(dore/kore) da  kke?
 0:20:20;06        whis/this   Cop Q
                  "Which one was it?/Was it this one?"

0:20:20;06    M:  ⌊((leans toward figure))

0:20:21;19    M:  kono ⌈sankakkee:: to  ⌈dokka    no san-
 0:20:24;12       this triangle     and somewhere LK tri-
                  "this triangle and some tri-"

0:20:21;23    M:       ⌊((repeats trace of the reference
 0:20:23;22       triangle))

0:20:23;28    M:                          ⌊((traces line between the
                  points at which the two parallel lines intersect with
                  the upper property boundary))

0:20:26;10    Ss: Hhh: ⌈hhh::
 0:20:26;25

0:20:26;26    T:       ⌊sono akai- akai(.) san⌈°kakkee°
 0:20:27;18             that red    red     triangle
                      "that red- red triangle"

0:20:28;06    M:                                ⌊a  kore desu ⌈ne.
 0:20:28;24                                       oh this Cop  FP
                                                  "Oh this one, right."

0:20:28;24    T:                                              ⌊u:n
 0:20:29;04                                                     uh.huh
                                                                "uh huh"

0:20:30;03    M:  ⌈WA:: (1.1) menseki ga::
 0:20:32;14       Top        area     S
                  "Their areas are::"

0:20:30;03    M:  ⌊((traces triangle starting from the point at
 0:20:32;12       which the r. parallel line intersects with the upper
                  boundary to the point where the l. parallel line
                  intersects the lower boundary then traveling to the
                  point at which the r. parallel line meets the lower
                  bound and finally returning to the original point
                  — see Figure 8b))
```

```
0:20:32;12    M:  ((holds pointer uncertainly over upper part of
0:20:34;01        figure))

0:20:33;09    T:  ⌈°kotchi kotchi° =
 0:20:33;22       this.side this.side
                  "This side this side"

00:20:33;09   T:  ⌊((heard walking toward Manabu))
 00:20:36;07

0:20:34;02    M:  = nan  desu ka? =
 0:20:34;27         what Cop  Q
                  "What is it?"

0:20:34;28    Ss: = ⌈HHH:: HHh: hhh:: hhh ⌈hhh
 0:20:36;29

0:20:34;28    M:    ⌊((waves tip of pointer over figure in agitated
 0:20:35;29         fashion))

0:20:36;23    T:                            ⌈((starting at the point
 0:20:38;26       at which the l. parallel line meets the upper
                  boundary, uses red chalk to highlight the line
                  segment connecting to the point at which the r.
                  parallel line intersects with the same boundary.
                  From there he draws a second line ending at the point
                  at which the l. parallel line intersects  with the
                  lower bound))

0:20:36;23    T:                            ⌊kotchi    no sankakkee. =
 0:20:38;02                                  this.side LK triangle
                                             "This side's triangle"
```

Locating the second triangle. As Livingston (1999) explained, the prover's task is ever and always one of finding the proof in the figure (p. 869). The power of the theorem rests in its ability to predict the area of all conforming triangles, that is, all triangles that have the same base and whose apexes lie on a line parallel to that base, will have equal areas. So in this case, to find the proof, Mr. Manabe must find a second conforming triangle within the solution figure, and here a problem arises. First, one of the students in the classroom calls into question his designation of height for the first triangle [0:20:17;16]. Mr. Manabe does not orient to this complaint, but instead begins, "This triangle a::nd" [0:20:15;17]. But, the thought is never completed. He interjects, "Right!" [0:20:20;06], but then seems unclear how to proceed. "Which one was it?" [0:20:20;06] he asks, while leaning in close to the solution figure. He then restarts with, "This triangle and some tri-" [0:20:21;19], but rather than introducing a new triangle, he retraces the one he had indicated earlier. The derivation, consequently, appears to be stalled out. If Manabe knew it before stepping to the board, a part of it, the identity of the second triangle, is now eluding him. The key to finding that triangle is to focus on the red line which may be difficult for his audience to see. The teacher tries to direct Mr. Manabe's attention to the triangle containing this line [0:20:26;26].

The third constructed line builds on the two described earlier. It connects the point at which the line forming the base of the first triangle intersects the lower property bound to the point where the constructed parallel line intersects the upper property boundary (shown as stroke #1 in Figure 8b). Mr. Manabe acknowledges the suggestion from the teacher, "Oh this one, right" [0:20:28;06], but still appears unsure regarding how to define a second triangle incorporating it. He attempts to complete the derivation step begun earlier with "this triangle and some tri-" [0:20:21;19] and states, "their areas are::" [0:20:30;03], while tracing a new triangle as shown in Figure 8b. Hayashi (2005) described how it is possible when producing an utterance in Japanese to use a "postpositional particle" as a hook for incorporating objects introduced earlier in the talk. In this case, we see Manabe using the topic marker *wa* in *Wa:: (1.1) menseki ga*:: ("Their areas are") to reference the pair of triangles previously introduced at [0:20:15;17] and again at [0:20:21;19].

The traced triangle (Figure 8b) has the same height as the triangle referenced earlier (Figure 8a), but does not share its base. It does not conform, therefore, to the requirements of the theorem. The instructor again seeks to help. He calls out, "This side this side" [0:20:33;09], but his suggestion is ambiguous and Manabe seeks further clarification [0:20:34;02]. The teacher advances to the board and, using more red chalk, makes bold the third constructed line and then similarly marks a segment of the upper property boundary [0:20:36;22]. As he does so, he clarifies that these lines are part of the triangle he was suggesting [0:20:36;23]. Manabe traces this triangle with the pointer (see Figure 8c) while finally completing the sentence begun initially at [0:20:17;04] and restarted at [0:20:21;19]. Repeating the designation just provided by the teacher [0:20:36;23], he refers to it as "this side's triangle" [0:20:38;02]. What begins in [0:20:15;17] as an "object-of-sorts with neither demonstrable sense nor reference" (Garfinkel, Lynch, & Livingston, 1981, p. 135), becomes, through the concerted efforts of the participants, a reference-able entity.

Excerpt 3e: Arriving at a conclusion.

```
0:20:38;02    M:  = ⌈kotchi   no  sankakkee to::
 0:20:39;25          this.side LK triangle   and
                     "and this side's triangle"

0:20:39;22    M:     ⌊((starting at the point at which the r.
 0:20:41;01        parallel line meets the upper boundary, he
                   twice traces the triangle formed by moving down the
                   diagonal line, up the left parallel line then
                   returning along the upper boundary to the beginning
                   point – see Figure 8c))
```

```
0:20:42;25    M:  menseki ga onaji na wake desu yo.
 0:20:44;12       area    S  same  Cop Nom Cop  FP
                  "their areas are the same."

0:20:44;27    M:  teehen to takasa ga onaji da  kara.
 0:20:46;10       base   and height S same  Cop because
                  "because their bases and heights are the same."


0:20:46;27    M:  da(h)kara:: (1.2) mazu  wa
 0:20:49:11       so                first Top
                  "So first"

0:20:50;14    M:  ⌈koko ni sen  o hikeru wake na  n   desu yo.
 0:20:51;17       here at line O draw   Nom  Cop Nom Cop  FP
                  "we can draw a line here."

0:20:50;14    M:  ⌊((traces new property line from bottom to top))
 0:20:51;04

0:20:52;29    M:  ⌈hai hai hai
 0:20:54;02       yes  yes  yes
                  "Yes yes yes."

0:20:53;02    M:  ⌊((retraces the property line from bottom to top))
 0:20:51;04
```

Arriving at a conclusion. As Livingston (1987) wrote, "The identifying orderliness of the work of a proof can be lost in too closely depicted particulars" (p. 109). A proof-account instructs a certain line of reasoning, but part of the art of producing a proof is in determining just how much instruction to provide. If too little, it may not be possible to follow the logic, but if too much is provided, the proof can appear pedantic, cluttered with trivial and unnecessary steps. What would serve as an adequate proof in a graduate seminar would obviously be quite different from what might be acceptable in a middle school geometry class and vice versa. Proof-accounts, in this way, are occasioned matters and entail a certain amount of "recipient design" (Sacks & Schegloff, 1979, p. 16). Drawing on the theorem, Mr. Manabe now notes that this pair of triangles have the same area [0:20:42;25] "because their bases and heights are the same" [0:20:44;27]. He then concludes, "So first" [0:20:46;27] "we can draw a line here" [0:20:50;14] while tracing the third constructed line. Manabe's *dakara mazu wa* is an enigmatic way of introducing a conclusion. Prefacing it with *dakara* is reasonable enough. In English, *so*, in sentence-initial positions, is often used to display inferential or logical connections to that that came before (Bolden, 2009; Schiffrin, 1987). His choice of *mazu wa*, however, presents a puzzle. This could be translated as "first" or "firstly" which might be a more sensible way to begin, rather than end a derivation and, as it happens, it was the way he began this derivation [0:19:50;11]. Conjunctions such as these play an important role in marking progress within a logical argument. By using *mazu wa* in an unconventional way, Mr. Manabe may have once again made his proof account a little less accessible. Nonetheless, the subsequent clause, "we can draw a line here" [0:20:50;14] represents both an articulation of the solution and the termination of his derivation. Its local sense derives from a gesture he performs with the pointer highlighting the red line on the solution figure. The referenced line is not just any line, but a candidate solution that satisfies the requirements of the property line problem. And the derivation just completed was produced to demonstrate this specifically.[13]

Excerpt 3f: "It's understandable enough, right?"

```
0:20:52;29    M:  ⌈hai hai ⌈hai
 0:20:54;02       yes  yes  yes
                  "Yes yes yes."

0:20:53;02    M:  ⌊((retraces the property line from bottom to top))
 0:20:51;04

0:20:53;09    S?:         ⌊un? (nani itte ⌈n mo)
 0:20:54;11                huh what say  Nom Q
                           "Huh? (what are you saying?)"

0:20:54;00    M:                                 ⌊maa  nani itte kka
  0:20:55;06                                      well what say  Q

                  wakannai      °(kedo).°
                  understand:Neg but

                 "Well I don't understand what I am saying, but"

0:20:55;11    Ss: ((laughter))
 0:20:56;17

0:20:56;08    M:  ((laughter))
 0:20:56;17

0:20:56;21    T:  juubun wakaru     yo ne.=
 0:20:57;24       enough understand FP FP
                  "It's understandable enough, right?"

0:20:57;25    M:  =a  wakarimasu ⌈ka.
 0:20:59;04        oh understand  Q
                  "Oh do you understand?"

0:20:58;17    T:                 ⌊wakannai      hito   iru? (0.9) (   )
 0:21:00;17                      understand:Neg person exist
                                 "Is there anyone who doesn't
                                  understand?"
```

[13]As an alternative way of appreciating Mr. Manabe's solution, try superimposing his various demonstrations at the board captured in Figure 8 upon the theorem illustration shown in Figure 3. Recall that in Figure 6 the teacher had the students change their frame of reference in order to more easily visualize how the theorem illustration related to the solution figures. You can accomplish the same by rotating Figure 8 counter-clockwise 90°. Mr. Manabe's first referenced triangle (Figure 8a) can then be seen as corresponding to triangle ACB in Figure 3 and the conforming triangle (Figure 8c) to ADB. Continuing on this basis, the line segment DB would represent the new property line demonstrated by Mr. Manabe at [0:20:50;14]. Because, by the stated theorem, triangles ADB and ACB have the same area, Chiba's property holdings must remain the same when using the newly straightened property line. And, if Chiba's holdings remain constant, it would stand to reason that the same is true of Bando's.

```
0:21:00;17    Ss: ((laughter))
 0:21:02;03

0:21:01;01    M:  ((laughter))
 0:21:02;03

0:21:02;14    T:  wakannai? =
 0:21:03;00       understand:Neg
                  "You don't understand?"

0:21:03;00    S?:   = watashi mo  wakannai.=
 0:21:03;25           I    also understand:Neg
                    "I don't understand, either."

0:21:03;26    T:  = >wakannai.<  ja   moo ikkai kotchigawa de
 0:21:05;29       understand:Neg then more once this.side  at
                  "You don't understand. Then once more at this side"

                  Ikeda  san ja   kotchi    de setsumeeshite.
                  surname Ms then this.side at explain
                  "Ms. Ikeda, please explain at this side."
```

"It's understandable enough, right?" Briefly studying that which he had just produced, Manabe reasserts the soundness of his account, saying "Yes" three times while retracing the line [0:20:53;13]. Despite this, his derivation may still have left too much tacit and unspoken. In response to Mr. Manabe's positive self-assessment ("Yes yes yes"), an unnamed student shouts, "Huh? (What are you saying?)" [0:20:53;09]. To this, Mr. Manabe cheerfully confesses "Well I don't understand what I am saying" [0:20:54;00]. This exchange is followed by laughter, both on the part of the class and from Mr. Manabe himself. The teacher then provides his own appraisal, "It's understandable enough, right?" [0:20:56;21]. Mr. Manabe responds, "Oh do you understand?" [0:20:59;04]. Without replying, the teacher turns to the class and asks, "Is there anyone who doesn't understand?" [0:20:58;17]. Several students confess that they too did not really follow Mr. Manabe's proof-account. His presentation, therefore, ends on an unhappy note.

Although we conclude our analysis here, the lesson was not quite finished. The problem had an alternative solution (see Figure 8d) and a second student, Ms. Ikeda, was invited to the board to present it (Mori & Koschmann, 2012). Following this, the teacher recapped the exercise and explained how the solutions presented by Mr. Manabe and Ms. Ikeda were related.

DISCOVERING A PROOF*

My use of the word "rigor" is adopted from conversational usage among mathematicians where it is used to refer to the witnessible (and, thereby, to the apparent, visible, recognizable) adequacy of a proof in demonstrating its own truthfulness, its own objectivity, its own accountability.

—Livingston (1983, p. 43)

Having presented this analysis, it would now be useful to bring what we have seen into alignment with Garfinkel's treatment of natural accountability and Livingston's notion of proof as a Lebenswelt pair. Mr. Manabe had been invited to the board to present a candidate solution to the posed property-line problem. His solution was visible from the outset in the form of the red chalk line, but to demonstrate its adequacy he was required to show how it could be derived from the stated conditions of the problem. This solution, then, was, in effect, the theorem to be proved. The various theorems that comprise Gödel's incompleteness proof predated Livingston's (1983) investigation. His thesis represented an effort to recover how a mathematician would work her way through this finished proof. In the case at hand, however, the proof is, at least in part, being constructed in the telling. Although we have referred to it as Mr. Manabe's presentation, he was clearly not the sole author. The proof-account was constructed collaboratively, building (or not building) on suggestions made by Mr. Manabe's classmates and incorporating crucial corrections from the teacher. The distinction between prover and proof-follower, as a result, was a little less clear-cut than in the example provided by Livingston.

We were able to discern elements of both classical and natural accountability being placed on display within the analyzed fragment. The construction of the solution figure, the organization of the presentation, the required repairs along the way all displayed an orientation to the accepted practices of the mathematics community and, in so doing, evidenced classical accountability (see Garfinkel, 2002, p. 188). Mr. Manabe's denial of understanding at the end of the fragment does not detract from this given that the accountability is a witnessible property of the proof-account itself.

The natural accountability of the proof* was to be found in the relationship between the proof-account and the practices of proving it describes. Garfinkel (2002) wrote, "The pair specifies [the] theorem as a lived organizational thing. The pair specifies [the] theorem as the practices of proving it" (p. 188).

But it is a specification of a peculiar sort, one that must be experienced to be appreciated. It is for this reason that we presented our analysis as a tutorial problem. Rather than describing Mr. Manabe's presentation in familiar geometric terms, it documented in terms of its worksite specifics. In so doing we hoped to make it possible for the reader to experience the discovering work that Livingston described as lying at the heart of mathematical practice. By joining Mr. Manabe's audience in their evaluation of the mathematical adequacy of his derivation, we come to discover the practices of proving that the proof-account described. Despite Mr. Manabe's unconventional start and finish, the various corrections from the audience, and his failure to recollect a crucial part of derivation, we find a coherent line of reason being developed here. There is, in short, a "natural analyzability and natural accountability" (Livingston, 1987, p. 112) to the presented derivation. We would take this to be the teacher's point when he proclaimed at the conclusion of Mr. Manabe's presentation, "It's understandable enough, right?"

One might reasonably ask, What is the value in making such a discovery? After all, we have argued that natural accountability is an indexical matter and, as a result, anything we might discover about it here is going to be irremediably tied the situated details of the situation at hand. But, as Livingston (1987) wrote,

> The practical methods of an ethnomethodological investigation are themselves made available and discovered through that investigation, they increasingly make available and rediscover the methods of a local production cohort in producing and managing the naturally organized ordinary activity in which that cohort is engaged. (p. 140)

Our analysis, therefore, not only is an analysis into *theirs*, but also provides guidance into how such analyses can be conducted in other settings. That is, it can be "mis-read" in the manner described by Garfinkel, as a set of instructions for looking and listening in a new way.

We find the efforts by Michaels et al. (2008) to focus attention on the importance of certain forms of classical accountability in classroom discourse to be useful. But we fear that if we consider only classical forms of accountability something vital will have been left from our appreciation of the lived work of teaching and learning mathematics. Garfinkel coined the name *natural* accountability for this missing something. Our ability to ultimately come to terms with the specification problem may well hinge upon the degree to which we develop an appreciation of both kinds of accountability in the classroom.

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